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Space-Time as a Micromorphic Continuum

Jan J. Sławianowski[1]

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We review generally-covariant Lagrangians for the field of linear coframes in an n-dimensional manifold. Discussed are Lagrangians invariant under the internal group $GL(n, \mathbb{R})$ and under its pseudo-Euclidean subgroups. It is shown that group spaces of semisimple Lie groups and certain of their modifications are natural vacuumlike solutions for all $GL(n, \mathbb{R})$-invariant models. In some sense the signature of space-time may be interpreted as a consequence of differential equations; the velocity of light is an integration constant.

Let M be an n-dimensional manifold—usual space-time or a higher-dimensional Kaluza-like universe. We shall consider a hypothetical physical system with degrees of freedom represented by the field of linear coframes $\mathbf{e} = (\ldots, e^K, \ldots)^T$, $K = 1, \ldots, n$. Interpreting $\mathbf{e}$ as an $\mathbb{R}^n$-valued Pfaff form, we shall denote the system of exterior differentials de^K by $d\mathbf{e}$. The dual contravariant frame will be denoted by $\tilde{\mathbf{e}} = (\ldots, e_K, \ldots)$. In terms of local coordinates x^i, $i = 1, \ldots, n$, the frames $\mathbf{e}$, $\tilde{\mathbf{e}}$ are analytically represented by matrices $[e^K_i]$, $[e^i_K]$, respectively; thus, locally, $e^K = e^K_i\, dx^i$, $e_K = e^i_K\, \partial/\partial x^i$. In elasticity such frame-based objects are known as *micromorphic continua*. By analogy to gauge theories, the anholonomic index K will be interpreted as an internal or isotopic index labeling basic fields within the n-dimensional multiplet.

The full linear group $GL(n, \mathbb{R})$ is a natural group of kinematic symmetries. It acts on fields $\mathbf{e}$ according to

$$L \in GL(n, \mathbb{R}): \quad L\mathbf{e} = L(\ldots, e^K, \ldots)^T := (\ldots, L^K_M e^M, \ldots)^T$$

I shall consider only *first-order* variational models with Lagrangians built algebraically of $\mathbf{e}$ and $d\mathbf{e}$, $L(\mathbf{e}, d\mathbf{e})$, L being a Weyl density of weight one. There is *a priori* a wide freedom of such Lagrangians; to make a reasonable

[1]Institute of Fundamental Technological Research, Polish Academy of Sciences, Warsaw, Poland.

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restriction, one must impose on them certain invariance conditions. First, of all, I consider only generally covariant models, $L[\varphi^* \mathbf{e}] = \varphi^* L[\mathbf{e}]$ modulo a total divergence; $\varphi \in \mathrm{Diff}\, M$ is an arbitrary diffeomorphism. Besides, it is natural to assume certain internal symmetry, $L[U\mathbf{e}] = L[\mathbf{e}]$ modulo a total divergence; $U \in G$, G being an as yet unspecified subgroup of $GL(n, \mathbb{R})$.

If one is not charged with any habits from general relativity and alternative theories of gravitation, then it seems that the best candidate for internal symmetry group is just $GL(n, \mathbb{R})$ itself—the natural group of kinematic symmetries. Thus, one should start with $GL(n, \mathbb{R})$-invariant models, and then investigate the hierarchy of models invariant under natural subgroups of $GL(n, \mathbb{R})$. It is natural to expect that the $GL(n, \mathbb{R})$-invariant models are good candidates for fundamental theories, whereas the restriction to subgroups $G \subset GL(n, \mathbb{R})$ should appear not on the level of fundamental equations, but rather on the level of solutions, when one investigates small vibrations about some fixed solutions, vacuums. This mechanism would be similar to the spontaneous symmetry breaking. I review Lagrangians invariant under $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, and under pseudo-Euclidean groups $O(k, n-k)$.

1. LAGRANGIANS INVARIANT UNDER $GL(n, \mathbb{R})$

Let us introduce $GL(n, \mathbb{R})$-invariant differential concomitants of the coframe $\mathbf{e}$. The basic quantity is the $\Gamma[\mathbf{e}]$-teleparallelism connection induced by $\mathbf{e}$ and uniquely defined by the demand $\nabla e^K = 0$, $K = 1, \ldots, n$. Analytically,

$$\Gamma^i_{jk} = e^i{}_M e^M{}_{j,k} = -e^M{}_j e^i{}_{M,k}$$

It is by definition curvature-free. Its torsion $S^i_{jk} = \Gamma^i_{[jk]}$ may be written as $S = -e_K \otimes de^K$ and is equivalent to the nonholonomy object of $\mathbf{e}$, as seen from the formulas

$$[e_K, e_L] = \gamma^M_{KL} e_M, \qquad de^K = \tfrac{1}{2}\gamma^K_{LM} e^M \wedge e^L$$
$$S = \tfrac{1}{2}\gamma^K_{LM} e_K \otimes e^L \otimes e^M$$

From S we can construct the following three-parameter family of twice covariant tensors:

$$\begin{aligned} T_{ij} &= A\gamma_{ij} + B\gamma_i\gamma_j + C\Gamma_{ij} \\ &= 4AS^a_{ib}S^b_{ja} + 4BS^a_{ia}S^b_{jb} + 4CS^b_{ab}S^a_{ij} \end{aligned}$$

A, B, C are real constants. This is the most general $GL(n, \mathbb{R})$-invariant tensor *quadratic in derivatives* of $\mathbf{e}$. The symmetric part

$$g_{ij} = A\gamma_{ij} + B\gamma_i\gamma_j$$

is the most natural candidate for the spatiotemporal metric tensor. Its second term is degenerate; thus, the main metriclike term is given by the *Killing tensor* γ_{ij}. If M is a Lie group and the e^K are basic right-invariant differential forms, then γ^K_{LM} are structure constants, and the nonholonomic components of the Killing tensor γ_{ij}, $\gamma_{AB}=\gamma^K_{AL}\gamma^L_{BK}$ coincide with the well-known Killing-Cartan metric coefficients of the corresponding Lie algebra.

There are also $GL(n,\mathbb{R})$-invariant scalars built of $\mathbf{e}$, $d\mathbf{e}$, e.g.,

$$\gamma_{ai}\gamma^{bj}\gamma^{ck}S^a_{bc}S^i_{jk}, \qquad \gamma^{ab}S^k_{ak}S^m_{bm}, \qquad \mathrm{Tr}[\gamma^{ac}\Gamma_{cb}]^p, \quad \text{etc.}$$

They are zero-degree homogeneous functions of derivatives.

$GL(n,\mathbb{R})$-invariant Lagrangians may be alternatively written in any of the following general forms, (Sławianowski, 1985):

$$L(S)=a(S)|\gamma|^{1/2}=b(S)|T|^{1/2} \tag{1}$$

where scalar functions a, b depend on S through the aforementioned scalars; $|\gamma|$ and $|T|$ denote the determinants of matrices of γ, T, respectively. Such Lagrangians are homogeneous of degree n in S. The simplest models are those with $a=\text{const}$ or $b=\text{const}$, i.e., $|\gamma|^{1/2}$ or $|T|^{1/2}$. The $GL(n,\mathbb{R})$-invariant models are strongly nonlinear; Lagrangians are never quadratic in derivatives and field equations are not quasilinear. The square-root structure resembles the Born-Infeld electrodynamics and leads one to expect a finite behavior of spherical solutions at $r=0$. The lack of gradient invariance makes such models similar to Mie's electrodynamics.

2. LAGRANGIANS INVARIANT UNDER $SL(n,\mathbb{R})\subset GL(n,\mathbb{R})$

They may be written in any of the following equivalent forms:

$$L=a(S,|\mathbf{e}|)|\gamma|^{1/2}=b(S,|\mathbf{e}|)|T|^{1/2}=c(S,|\mathbf{e}|)|\mathbf{e}| \tag{2}$$

where $|\mathbf{e}|=\det[e^K_i]$, and a, b, c are functions of $\mathbf{e}$ and basic invariants of S. I do not consider such models in this paper.

3. LAGRANGIANS INVARIANT UNDER $O(k,n-k)\subset GL(n,\mathbb{R})$

To describe them we introduce another concomitant of $\mathbf{e}$, namely the so-called *Dirac-Einstein metric* $h[\mathbf{e}]$:

$$h[\mathbf{e}]:=\eta_{KL}e^K\otimes e^L, \quad \text{i.e.,} \quad h_{ij}=\eta_{KL}e^K_i e^L_j$$

where

$$[\eta_{KL}]=\mathrm{diag}\Big(\underbrace{1,\ldots,1}_{k};\ \underbrace{-1,\ldots,-1}_{n-k}\Big)$$

In tetrad-based Lorentz-invariant theories of gravitation $h[\mathbf{e}]$ plays the role of the spatiotemporal metric tensor. Its signature is introduced by hand through η. The $h[\mathbf{e}]$ is constant under the e-parallelism, $\nabla h[\mathbf{e}]=0$, and orthonormal in the η-sense. Note that the Killing tensor in general is not ∇-parallel. $O(k, n-k)$ is the largest subgroup of $GL(n, \mathbb{R})$ preserving $h[\mathbf{e}]$,

$$h[U\mathbf{e}]=h[\mathbf{e}] \quad \text{iff} \quad U\in O(k, n-k) \tag{3}$$

$O(k, n-k)$-invariant Lagrangians may be written as follows:

$$L(S, h)=a(S, h)|\gamma|^{1/2}=b(S, h)|T|^{1/2}=c(S, h)|h|^{1/2}=c(S, h)|\mathbf{e}| \tag{4}$$

a, b, c are built of basic invariants of S. The most convenient form is that based on the shape factor c. There are models quadratic in derivatives; they are linear combinations of three invariants,

$$L=c_1L_1+c_2L_2+c_3L_3=(c_1J_1+c_2J_2+c_3J_3)|\mathbf{e}| \tag{5}$$

$$J_1:=h_{ai}h^{bj}h^{ck}S^a_{bc}S^i_{jk}$$

$$J_2:=h^{ab}S^m_{an}S^n_{bm}=\tfrac{1}{4}h^{ab}\gamma_{ab}$$

$$J_3:=h^{ab}S^k_{ak}S^m_{bm}=\tfrac{1}{4}h^{ab}\gamma_a\gamma_b$$

Such Lagrangians, with $k=1$ and $n=4$, are used in metric-teleparallel theories of gravitation (Hehl *et al.*, 1980; Kopczyński, 1982). Einstein's theory corresponds to the special case $c_1:c_2:c_3=1:2:(-4)$; it is locally Lorentz-invariant, i.e., U in (3) may depend on x^i. There is a wide family of models (5) as compatible with experiment as Einstein's theory, but more convincing from the point of view of gauge philosophy. Nonquadratic models (4) were suggested as a way out of singularities of Einstein's theory (Möller, 1978).

4. GROUP SPACES AS FUNDAMENTAL SOLUTIONS

I now discuss the general form of field equations for $O(k, n-k)$ and $GL(n, \mathbb{R})$ models. I use tensor densities

$$H^{ij}_k=-H^{ji}_k:=\frac{\partial L}{\partial S^k_{ij}}, \qquad Q^{ij}=Q^{ji}:=\frac{\partial L}{\partial h_{ij}}$$

The field equations may be written as

$$K^j_i:=\nabla_kH^{jk}_i+2S^m_{mk}H^{jk}_i-2h_{ik}Q^{kj}=0 \tag{6}$$

and in the case of $GL(n, \mathbb{R})$-invariant models simply as

$$K^j_i=\nabla_kH^{jk}_i+2S^m_{mk}H^{jk}_i=0 \tag{7}$$

∇ denotes the covariant differentiation in the sense of $\Gamma[\mathbf{e}]$. It will be convenient to use the form

$$G^{jkbc}_{ia}\nabla_kS^a_{bc}+2S^m_{mk}H^{jk}_i=2h_{ik}Q^{kj} \tag{8}$$

where, using obvious abbreviations, $G := \partial H/\partial S = \partial^2 L/\partial S\, \partial S$. The generally covariant equations (6) are overdetermined and involve n redundant variables to be fixed by coordinate conditions. Their strong nonlinearity prevents us from performing a Dirac analysis and deciding how strong are the integrability conditions. All that is evident is that the equations $K^0_a = 0$ belong to secondary constraints; the zero label refers to the "time" variable. *A priori* it is not clear whether equations (6), (7) are consistent. Nevertheless, one can constructively show they are, because there exist geometrically distinguished vacuumlike solutions. $O(k, n-k)$-invariant models provide a hint for guessing them. Namely, it is clear that for quadratic Lagrangians (5) every holonomic coframe is a solution. The corresponding metric $h[\mathbf{e}]$ is flat, $d\mathbf{e} = 0$ and $S = 0$. Such solutions are classical vacuums in the sense they are homogeneous situations as nonexcited physically as possible, just as $\psi = 0$ is the classical vacuum of the Klein–Gordon field. If $\mathbf{e}$ is holonomic, the contravariant vectors e_K, $K = 1, \ldots, n$, generate the local Abelian group of transformations acting transitively and freely in open domains of M. Thus, (M, e) is, at least locally, an Abelian group space. Do such *Abelian vacuums* exist for nonquadratic models $L(S, h)$? It is easy to formulate the following sufficient condition.

For a general $O(k, n-k)$-invariant model $L(S, h)$, Abelian vacuums, i.e., solutions given by holonomic coframes, exist if:

(i) $Q(0, h) = \partial L/\partial h(0, h) = 0$, i.e., Q vanishes with S

(ii) $H(0, h) = \partial L/\partial S(0, h)$, $G(0, h) = \partial^2 L/\partial S\, \partial S(0, h)$ are finite and one-valued

This follows from equation (8). For $GL(n, \mathbb{R})$ models the condition (i) is trivially satisfied, but (ii) fails. Indeed, for $S = 0$, Lagrangians (1) have a differential singularity $\sqrt{0}$, and both H and G are singular. Thus, the left-hand side of (8) is not well-defined. One could try to argue that it vanishes for $S = 0$ if $n > 2$, because it consists of two terms which are homogeneous of degree $n-2$ and $n-1$ in S. This would be mathematically artificial, though. Besides, γ vanishes then and the formalism has no metrical interpretation.

There exist, however, *semisimple vacuums*. Every semisimple Lie form, i.e., a coframe for which contravariant vectors e_K span a semisimple Lie algebra, is a solution of (7) in principle for any model $L(S)$. Nonholonomy coefficients γ^K_{LM} are then structure constants, S is parallel, $\nabla S = 0$, and traceless, because structure constants of semisimple Lie algebras are traceless. Thus, if a in (1) is smooth at $S = 0$, then $H(0)$ and $G(0)$ are finite and equation (8) is evidently satisfied. The vector fields e_K generate a semisimple Lie group whose action on M is transitive and free. Thus, canonical forms on semisimple Lie groups are universal solutions for all $GL(n, \mathbb{R})$-invariant models. For such solutions the Killing metric $\gamma[\mathbf{e}]$ is parallel, $\nabla\gamma[\mathbf{e}]=0$. It has a $2n$-dimensional isotropy group generated by the e_K and by fields e_K^* conjugate to the e_K with respect to a fixed point of M identified with the group identity (if the e_K are right invariant, then the e_K are left invariant). Small excitations of such solutions feel the effective background of an absolute-like geometry created by the vacuum solutions. This background is a pseudo-Riemann–Cartan space.

Similar ideas of Lie group spaces as vacuums of field-theoretic models based on differential forms were formulated in Toller (1980), D'Adda *et al.* (1985) and Halpern (1984). Unfortunately, there are no semisimple four-dimensional Lie algebras; thus, either we have to give up the nice idea of group-space vacuums or save them in higher-dimensional Kaluza-like worlds. Fortunately, it turns out that certain grouplike vacuums exist also in dimensions "semisimple plus one," and thus also for $n=4$. They are given by appropriately deformed canonical forms of trivial central extensions of $(n-1)$-dimensional semisimple Lie groups. Moreover, in some sense such vacuums explain the normal-hyperbolic signature as a consequence of differential equations.

From now on I shall use the relativistic convention of Latin and Greek indices; thus, k, $K=1,\ldots,n-1$, and λ, $\Lambda=0,1,\ldots,n-1$. Let us introduce an auxiliary coframe $E=(E^0,\ldots,E^K,\ldots)^T$ such that

$$dE^0=0, \qquad dE^K=\tfrac{1}{2}C^K_{LM}E^M\wedge E^L$$

C^M_{KL} denotes the structure constants of a semisimple Lie algebra; thus $[C_{KL}]:=[C^A_{KB}C^B_{LA}]$ is nonsingular. In adapted coordinates $(\ldots,x^\mu,\ldots)=(t,\ldots,x^i,\ldots)$, we have $E^0=dt$, $E^K=E^K_i(x^j)\,dx^i$. The duals E_Σ span a nonsemisimple Lie algebra with one-dimensional center. Let us introduce a new coframe $\mathbf{e}=(e^0,\ldots,e^K,\ldots)^T$, where

$$e^0=E^0, \quad e^K=\frac{1}{\lambda}E^K, \quad \text{i.e.,} \quad e_0=E_0, \quad e_K=\lambda E_K$$

and $E_K\lambda=e_K\lambda=0$. In adapted coordinates

$$e^0=dt, \qquad e^K=\frac{1}{\lambda(t)}E^K_i(x^j)\,dx^i$$

Such coframes will be called *Lie-developing forms.* They are called *normal* if the E_K span a compact Lie algebra. The Killing metric $\gamma[\mathbf{e}]$ is nonsingular if and only if λ has no critical points,

$$\gamma[\mathbf{e}]=(n-1)\left(\frac{d\log\lambda}{dt}\right)^2E^0\otimes E^0+C_{KL}E^K\otimes E^L$$

Locally,

$$\gamma[\mathbf{e}] = (n-1)d \log \lambda \otimes d \log \lambda + 4S^k_{im}S^m_{jk}\, dx^i \otimes dx^j$$

If the Lie algebra spanned by the E_K is compact, then $\gamma[\mathbf{e}]$ is normal-hyperbolic, e_0 is timelike, and the e_K are spacelike and orthogonal to e_0 in the $\gamma[\mathbf{e}]$ sense. The proper time T measured along integral curves of e_0 by $\gamma[\mathbf{e}]$ is an affine function of $\log \lambda$, $T = (1/\alpha) \log \lambda + \text{const}$. Putting $\lambda = \delta \exp(\alpha t)$, we have

$$\gamma[\mathbf{e}] = (n-1)\alpha^2\, dt \otimes dt + \gamma_{ij}(x^k)\, dx^i \otimes dx^j, \qquad \gamma_{ij} = 4S^k_{im}S^m_{jk}$$

$\gamma[\mathbf{e}]$ is stationary and static, t and x^i are, respectively, time and spatial coordinates, $c = |\alpha|(n-1)^{1/2}$ is the velocity of light, and locally $M = \mathbb{R} \times G$; $\mathbb{R}$ represents the time axis, and G the compact $(n-1)$-dimensional space. Although $\mathbf{e}$ itself depends exponentially on time, all $GL(n, \mathbb{R})$-invariant quantities built of $\mathbf{e}$ are time-independent, just like the Killing metric $\gamma[\mathbf{e}]$. There is a $(2n-1)$-parameter isotropy group generated by E_0, the E_K, and their conjugates E^*_K.

It turns out that any Lie-developing coframe on M with an arbitrary expansion factor λ is a solution of (7) for any $GL(n, \mathbb{R})$-invariant Lagrangian $L(S)$. Such solutions are *developing non-Abelian vacuums.* Their $GL(n, \mathbb{R})$-invariant characteristics are static and stationary; nevertheless, some kind of expansion and arrow of time is hidden in internal variables. For example, let us construct the Dirac–Einstein metric with $\eta_{KL} = C_{KL}$, $\eta_{K0} = 0$, $\eta_{00} = \beta^2$, $\lambda = \delta e^{\alpha t}$:

$$h[\mathbf{e}, \eta] = \beta^2\, dt \otimes dt + \delta^{-2} \exp(-2\alpha t) C_{KL} E^K \otimes E^L$$

The $(n-1)$-dimensional spatial part of the metric undergoes a de Sitter-like expansion with the factor $\exp(-2\alpha t)$. Recall that in standard theories, fermion fields interact with the gravitational tetrad just through $h[\mathbf{e}]$. Thus, test matter injected into a Lie-developing vacuum will be seen as expanding in spite of the static character of the Killing metric $\gamma[\mathbf{e}]$. This leads to obvious speculations about the red-shift and escaping of galaxies. However it is a mere speculation; I have yet no indication of whether the $GL(n, \mathbb{R})$ Lagrangians may be used as alternative models of macroscopic gravitation (my motivation was instead microscopic). I have no convincing results concerning stationary spherical solutions about a point source.

Let me finish with another speculation, concerning the signature and space-time dimension. They suggest that $GL(4, \mathbb{R})$ models might be part of the proper theory of space-time. Dimensions 1 and 2 are impossible: if $n = 1$, then $S = 0$; if $n = 2$, $\det[\gamma_{ij}] = \det[T_{ij}] = 0$, thus everything would be trivial. If $n = 3$, there are no Lie-developing vacuums, because there are no semisimple two-dimensional Lie algebras. There are three-dimensional Lie solutions corresponding to the simple algebras $su(2)$, $sl(2, \mathbb{R})$. However, in the first case the Killing metric is elliptic; for $sl(2, \mathbb{R})$ it is normal-hyperbolic, but there are closed timelike lines corresponding to the compact dimension in $SL(2, \mathbb{R})$. If $n=4$, there are no Lie solutions; nevertheless, there are two Lie-developing vacuums corresponding to Lie algebras $su(2)$, $sl(2, \mathbb{R})$. In the first case the Killing tensor has signature $(+---)$ and locally $M \simeq \mathbb{R}\times S^3 \simeq \mathbb{R}\times SU(2)$; $\mathbb{R}$ is the time axis and S^3 the 3-dimensional finite space. In the second case γ has the reversed signature $(+++-)$ and M locally coincides with $\mathbb{R}\times SL(2, \mathbb{R})$. However, the $\mathbb{R}$ component is now spacelike and there are closed timelike lines.

Thus, if we assume the normal-hyperbolic signature of γ as a physical feature of solutions **e**, then $n=4$ is the lowest nontrivial dimension. Conversely, assuming $n=4$, we see that the normal-hyperbolic signature is a property of the most natural vacuumlike solutions. Roughly speaking, it is implied by differential equations. The velocity of light, $c=|\alpha|(n-1)^{1/2}$, is not a primitive constant introduced by hand; it is a parameter, a kind of integration constant characterizing a given solution. For $n>4$ our treatment could be attempted to explain dynamically the gauge groups and spatio-temporal fibrations of Kaluza-like worlds in terms of particular solutions for n-legs, without losing n-dimensional general covariance.

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